

GENERIC INITIAL IDEALS AND GRADED ARTINIAN LEVEL ALGEBRAS NOT HAVING THE WEAK-LEFSCHETZ PROPERTY

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ABSTRACT. We find a sufficient condition that \mathbf{H} is not level based on a reduction number. In particular, we prove that a graded Artinian algebra of codimension 3 with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_{d-1} > h_d = h_{d+1})$ cannot be level if $h_d \leq 2d + 3$, and that there exists a level O-sequence of codimension 3 of type \mathbf{H} for $h_d \geq 2d + k$ for $k \geq 4$. Furthermore, we show that \mathbf{H} is not level if $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$, and also prove that any codimension 3 Artinian graded algebra $A = R/I$ cannot be level if $\beta_{1,d+2}(\text{Gin}(I)) = \beta_{2,d+2}(\text{Gin}(I))$. In this case, the Hilbert function of A does not have to satisfy the condition $h_{d-1} > h_d = h_{d+1}$.

Moreover, we show that every codimension n graded Artinian level algebra having the Weak-Lefschetz Property has the strictly unimodal Hilbert function having a growth condition on $(h_{d-1} - h_d) \leq (n-1)(h_d - h_{d+1})$ for every $d > \theta$ where

$$h_0 < h_1 < \dots < h_\alpha = \dots = h_\theta > \dots > h_{s-1} > h_s.$$

In particular, we find that if A is of codimension 3, then $(h_{d-1} - h_d) < 2(h_d - h_{d+1})$ for every $\theta < d < s$ and $h_{s-1} \leq 3h_s$, and prove that if A is a codimension 3 Artinian algebra with an h -vector $(1, 3, h_2, \dots, h_s)$ such that

$$h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0 \quad \text{and} \quad \text{soc}(A)_{d-1} = 0$$

for some $r_1(A) < d < s$, then $(I_{\leq d+1})$ is $(d+1)$ -regular and $\dim_k \text{soc}(A)_d = h_d - h_{d+1}$.

1. INTRODUCTION

Let $R = k[x_1, \dots, x_n]$ be an n -variable polynomial ring over an infinite field with characteristic 0. In this article, we shall study Artinian quotients $A = R/I$ of R where I is a homogeneous ideal of R . These rings are often referred to as standard graded algebras. Since $R = \bigoplus_{i=0}^{\infty} R_i$ (R_i : the vector space of dimension $\binom{i+(n-1)}{n-1}$ generated by all the monomials in R having degree i) and $I = \bigoplus_{i=0}^{\infty} I_i$, we get that

$$A = R/I = \bigoplus_{i=0}^{\infty} (R_i/I_i) = \bigoplus_{i=0}^{\infty} A_i$$

is a graded ring. The numerical function

$$\mathbf{H}_A(t) := \dim_k A_t = \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the ring A .

Given an O-sequence $\mathbf{H} = (h_0, h_1, \dots)$, we define the *first difference* of \mathbf{H} as

$$\Delta \mathbf{H} = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \dots).$$

If I is a homogeneous ideal of R of height n , then $A = R/I$ is an *Artinian k -algebra*, and hence $\dim_k A < \infty$. We associate to the graded algebra A a vector of nonnegative integers which is an $(s+1)$ -tuple, called the *h-vector* of A and denoted by

$$h(A) = (h_0, h_1, \dots, h_s),$$

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where $h_i = \dim_k A_i$. Thus we can write $A = k \oplus A_1 \oplus \cdots \oplus A_s$ where $A_s \neq 0$. We call s the *socle degree* of A . The *socle* of A is defined by the annihilator of the maximal homogeneous ideal, namely

$$\text{ann}_A(m) := \{a \in A \mid am = 0\} \quad \text{where} \quad m = \sum_{i=1}^s A_i.$$

Moreover, an h -vector (h_0, h_1, \dots, h_s) is called

$$\begin{aligned} &\text{unimodal if} & h_0 \leq \cdots \leq h_t = \cdots = h_\ell \geq \cdots \geq h_s, \\ &\text{strictly unimodal if} & h_0 < \cdots < h_t = \cdots = h_\ell > \cdots > h_s. \end{aligned}$$

A graded Artinian k -algebra $A = \bigoplus_{i=0}^s A_i$ ($A_s \neq 0$) is said to have the *Weak Lefschetz Property* (WLP for short) if there is an element $L \in A_1$ such that the linear transformations

$$A_i \xrightarrow{\times L} A_{i+1}, \quad 1 \leq i \leq s-1,$$

which is defined by multiplication by L , are either injective or surjective. This implies that the linear transformations have maximal ranks for every i . In this case, we call L a *Lefschetz element*.

A monomial ideal I in R is *stable* if the monomial

$$\frac{x_j w}{x_{m(w)}}$$

belongs to I for every monomial $w \in I$ and $j < m(w)$ where

$$m(u) := \max\{j \mid a_j > 0\}$$

for $u = x_1^{a_1} \cdots x_n^{a_n}$. Let S be a subset of all monomials in $R = \bigoplus_{i \geq 0} R_i$ of degree i . We call S a *Borel fixed set* if

$$u = x_1^{a_1} \cdots x_n^{a_n} \in S, \quad a_i > 0, \quad \text{implies} \quad \frac{x_i u}{x_j} \in S$$

for every $1 \leq i \leq j \leq n$.

A monomial ideal I of R is called a *Borel fixed ideal or strongly stable ideal* if the set of all monomials in I_i is a Borel set for every i . There are two Borel fixed monomial ideals canonically attached to a homogeneous ideal I of R : the generic initial ideal $\text{Gin}(I)$ with respect to the reverse lexicographic order and the lex-segment ideal I^{lex} . The ideal I^{lex} is defined as follows. For the vector space I_d of forms of degree d in I , one defines $(I^{\text{lex}})_d$ to be the vector space generated by largest, in the lexicographical order, $\dim_k(I_d)$ monomials of degree d . By construction, I^{lex} is a strongly stable ideal and it only depends on the Hilbert function of I .

In case of the generic initial ideal, it has been proved that generic initial ideals are Borel fixed in characteristic zero by Galligo [13], and then generalized by Bayer and Stillman to every characteristic [2].

In [1], they gave some geometric results using generic initial ideals for the degree reverse lexicographic order, which improved a well known result of Bigatti, Geramita, and Migliore concerning geometric consequences of maximal growth of the Hilbert function of the Artinian reduction of a set of points in [6]. In [15], they gave a homological reinterpretation of a level Artinian algebra and explained the combinatorial notion of Cancellation of Betti numbers of the minimal free resolution of the lex-segment ideal associated to a given homogeneous ideal. We shall explain the new result when we carry out the analogous result using the generic initial ideal instead of the lex-segment ideal. We find some new results on the maximal growth of the difference of Hilbert function in degree d larger than the reduction number $r_1(A)$ if there is no socle element in degree $d-1$ using some recent result in [1]. As an application, we give the condition if some O-sequence is “either level or non-level sequences of Artinian graded algebras with the WLP.”

Let \mathcal{F} be the graded minimal resolution of R/I , i.e.,

$$\mathcal{F} : 0 \rightarrow \mathcal{F}_n \rightarrow \mathcal{F}_{n-1} \rightarrow \cdots \rightarrow \mathcal{F}_1 \rightarrow R \rightarrow R/I \rightarrow 0.$$

We can write

$$\mathcal{F}_i = \bigoplus_{j=1}^{\gamma_i} R^{\beta_{ij}}(-\alpha_{ij})$$

where $\alpha_{i1} < \alpha_{i2} < \cdots < \alpha_{i\gamma_i}$. The numbers α_{ij} are called the *shifts* associated to R/I , and the numbers β_{ij} are called the *graded Betti numbers* of R/I . For I as above, the *Betti diagram* of R/I is a useful device to encode the graded Betti numbers of R/I (and hence of I). It is constructed as follows:

$$\begin{array}{ccccccccc} & & 0 & & 1 & & \cdots & & n-1 \\ 0 & & \left(\begin{array}{cccccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t & 0 & \beta_{0,t+1} & \beta_{1,t+2} & * & \beta_{n-1,t+n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d-2 & 0 & \beta_{0,d-1} & \beta_{1,d} & * & \beta_{n-1,d-2+n} \\ d-1 & 0 & \beta_{0,d} & \beta_{1,d+1} & * & \beta_{n-1,d-1+n} \\ d & 0 & \beta_{0,d+1} & \beta_{1,d+2} & * & \beta_{n-1,d+n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right) \end{array}$$

When we need to emphasize the ideal I , we shall use $\beta_{i,j}(I)$ for $\beta_{i,j}$.

Now, we recall that if the last free module of the minimal free resolution of a graded ring A with Hilbert function \mathbf{H} is of the form $\mathcal{F}_n = R^\beta(-s)$ for some $s > 0$, then Hilbert function \mathbf{H} and a graded ring A are called *level*. For a special case, if $\beta = 1$, then we call a graded Artinian algebra A *Gorenstein*. In [32], Stanley proved that any graded Artinian Gorenstein algebra of codimension 3 is unimodal. In fact, he proved a stronger result than unimodality using the structure theorem of Buchsbaum and Eisenbud for the Gorenstein algebra of codimension 3 in [8]. Since then, the graded Artinian Gorenstein algebras of codimension 3 have been much studied (see [9], [15], [16], [20], [21], [27], [28], [31], [33]). In [3], Bernstein and Iarrobino showed how to construct non-unimodal graded Artinian Gorenstein algebras of codimension higher than or equal to 5. Moreover, in [7], Boij and Laksov showed another method on how to construct the same graded Artinian Gorenstein algebras. Unfortunately, it has been unknown if there exists a graded non-unimodal Gorenstein algebra of codimension 4. For unimodal Artinian Gorenstein algebras of codimension 4, how to construct some of them using the link-sum method has been shown in [31]. It has been also shown in [16] and [20] how to obtain some of unimodal Artinian Gorenstein algebras of any codimension n (≥ 3). An SI-sequence is a finite sequence of positive integers which is symmetric, unimodal and satisfies a certain growth condition. In [28], Migliore and Nagel showed how to construct a reduced, arithmetically Gorenstein configuration G of linear varieties of arbitrary dimension whose Artinian reduction has the given SI-sequence as Hilbert function and has the Weak Lefschetz Property. For graded Artinian level algebras, it has been recently studied (see [3], [5], [7], [10], [15], [17], [27], [33], [34]). In [15], they proved the following result. Let

$$(1.1) \quad \mathbf{H} : h_0 \ h_1 \ \cdots \ h_{d-1} \ h_d \ h_d \ \cdots$$

with $h_{d-1} > h_d$. If $h_d \leq d+1$ with any codimension h_1 , then \mathbf{H} is *not* level.

In [33], F. Zanello constructed a non-unimodal level O-sequence of codimension 3 as follows:

$$\mathbf{H} = (h_0, h_1, \dots, h_d, t, t, t+1, t, t, \dots, t+1, t, t)$$

where the sequence $t, t, t+1$ can be repeated as many times as we want. Thus there exists a graded Artinian level algebra of codimension 3 of type in equation (1.1) which does not have the WLP.

In Section 2, preliminary results and notations on lex-segment ideals and generic initial ideals are introduced. In Section 3, we show that any codimension n graded Artinian level algebra A having the WLP has the Hilbert function which is strictly unimodal (see Theorem 3.6). In particular, we prove that if A has the Hilbert function such that

$$h_0 < h_1 < \cdots < h_{r_1(A)} = \cdots = h_\theta > \cdots > h_{s-1} > h_s,$$

then $h_{d-1} - h_d \leq (n-1)(h_d - h_{d+1})$ for every $\theta < d \leq s$ (see Theorem 3.6). Furthermore, we show that if A is of codimension 3, then $h_{d-1} - h_d < 2(h_d - h_{d+1})$ for every $\theta < d < s$ and $h_{s-1} \leq 3h_s$ (see Theorem 3.23). We also prove that if A is a codimension 3 Artinian graded algebra with socle degree s and

$$\beta_{1,d+2}(\text{Gin}(I)) = \beta_{2,d+2}(\text{Gin}(I)) > 0$$

for some $d < s$, then A cannot be level (see Theorem 3.14). Moreover, if $A = R/I$ is a codimension 3 Artinian graded algebra with an h -vector $(1, 3, h_2, \dots, h_s)$ such that $h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0$ for some $r_1(A) < d < s$ and $\text{soc}(A)_{d-1} = 0$, then $(I_{\leq d+1})$ is $(d+1)$ -regular and $\dim_k \text{soc}(A)_d = h_d - h_{d+1}$ (see Theorem 3.19).

One of the main topics of this paper is to study O-sequences of type in equation (1.1) and find an answer to the following question.

Question 1.1. Let \mathbf{H} be as in equation (1.1) with $h_1 = 3$. What is the minimum value for h_d when \mathbf{H} is level?

Finally in Section 4, we show that if R/I is a graded Artinian algebra of codimension 3 having Hilbert function \mathbf{H} in equation (1.1) and $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$, then R/I is **not** level, i.e., \mathbf{H} cannot be level (see Theorem 4.5). Furthermore, we prove that any O-sequence \mathbf{H} of codimension 3 in equation (1.1) cannot be level when $h_d \leq 2d+3$ and there exists a level O-sequence of codimension 3 of type in equation (1.1) having $h_d \geq 2d+k$ for every $k \geq 4$ (see Theorem 4.1, Proposition 4.9 and Remark 4.10), which is a complete answer to Question 1.1.

A computer program CoCoA was used for all examples in this article.

2. SOME PRELIMINARY RESULTS

In this section, we introduce some preliminary results and notations on lex-segment ideals and generic initial ideals.

Theorem 2.1 ([1], [2], [19]). *Let L be a general linear form and let $J = (I + (L))/(L)$ be considered as a homogeneous ideal of $S = k[x_1, \dots, x_{n-1}]$. Then*

$$\text{Gin}(J) = (\text{Gin}(I) + (x_n))/(x_n).$$

For a homogeneous ideal $I \subset R$ there exists a flat family of ideals I_t with $I_0 = \text{in}(I)$ (the initial ideal of I) and I_t canonically isomorphic to I for all $t \neq 0$ (this implies that $\text{in}(I)$ has the same Hilbert function as the one of I). Using this result, we get the following Theorem:

Theorem 2.2 (The Cancelation Principle, [1], [19]). *For any homogeneous ideal I and any i and d , there is a complex of $k \cong R/m$ -modules V_\bullet^d such that*

$$\begin{aligned} V_i^d &\cong \text{Tor}_i^R(\text{in}(I), k)_d \\ H_i(V_\bullet^d) &\cong \text{Tor}_i^R(I, k)_d. \end{aligned}$$

Remark 2.3. One way to paraphrase this Theorem is to say that the minimal free resolution of I is obtained from that of $\text{in}(I)$, the *initial ideal* of I , by canceling some adjacent terms of the same degree.

Theorem 2.4 (Eliahou-Kervaire, [11]). *Let I be a stable monomial ideal of R . Denote by $\mathcal{G}(I)$ the set of minimal (monomial) generators of I and $\mathcal{G}(I)_d$ the elements of $\mathcal{G}(I)$ having degree d . Then*

$$\beta_{q,i}(I) = \sum_{T \in \mathcal{G}(I)_{i-q}} \binom{m(T) - 1}{q}.$$

This theorem gives all the graded Betti numbers of the lex-segment ideal and the generic initial ideal just from an intimate knowledge of the generators of that ideal. Since the minimal free resolution of the ideal of a k -configuration in \mathbb{P}^n is extremal ([16], [18]), we may apply this result to those ideals. It is an immediate consequence of the Eliahou–Kervaire theorem that if I is a lex-segment ideal, a generic initial ideal, or the ideal of a k -configuration in \mathbb{P}^n which has no generators in degree d , then $\beta_{q,i} = 0$ whenever $i - q = d$.

Remark 2.5. Let I be any homogeneous ideal of $R = k[x_1, \dots, x_n]$ and $J = \text{Gin}(I)$. Then, by Theorem 2.2, we have

$$\beta_{q,i}(I) \leq \beta_{q,i}(J).$$

In particular, if $\beta_{q,i}(J) = 0$, then $\beta_{q,i}(I) = 0$.

Let I be a homogeneous ideal of $R = k[x_1, \dots, x_n]$ such that $\dim(R/I) = d$. In [23], they defined the s -reduction number $r_s(R/I)$ of R/I for $s \geq d$ and have shown the following theorem.

Theorem 2.6 ([1], [23]). *For a homogeneous ideal I of R ,*

$$r_s(R/I) = r_s(R/\text{Gin}(I)).$$

If I is a Borel fixed monomial ideal of $R = k[x_1, \dots, x_n]$ with $\dim(R/I) = n - d$, then we know that there are positive numbers a_1, \dots, a_d such that $x_i^{a_i}$ is a minimal generator of I . In [23], they have also proved that if a monomial ideal I is strongly stable, then

$$r_s(R/I) = \min\{\ell \mid x_{n-s}^{\ell+1} \in I\}.$$

Furthermore, the following useful lemma has been proved in [1].

Lemma 2.7 (Lemma 2.15, [1]). *For a homogeneous ideal I of R and for $s \geq \dim(R/I)$, the s -reduction number $r_s(R/I)$ can be given as the following:*

$$\begin{aligned} r_s(R/I) &= \min\{\ell : x_{n-s}^{\ell+1} \in \text{Gin}(I)\} \\ &= \min\{\ell : \text{Hilbert function of } R/(I + J) \text{ vanishes in degree } \ell + 1\} \end{aligned}$$

where J is generated by s general linear forms of R .

For a homogeneous ideal I of $R = k[x_1, \dots, x_n]$, we recall that I^{lex} is a lex-segment ideal associated to I . In Section 4, we shall use the following two useful lemmas.

Lemma 2.8. *Let I be a homogeneous ideal of $R = k[x_1, \dots, x_n]$ and let $\bar{I} = (I_{\leq d+1})$ for some $d > 0$. Then,*

- (a) $\beta_{i,j}(I) \leq \beta_{i,j}(\text{Gin}(I)) \leq \beta_{i,j}(I^{\text{lex}})$ for all i, j .
- (b) $\beta_{0,d+2}(\bar{I}^{\text{lex}}) = \beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)$,
- (c) $\beta_{0,d+2}(\text{Gin}(\bar{I})) = \beta_{0,d+2}(\text{Gin}(I)) - \beta_{0,d+2}(I)$.

Proof. (a) The first inequality can be proved by Theorem 2.2. The second one is directly obtained from the theorem of Bigatti, Hulett, and Pardue ([4], [24], and [29]).

(b) First note that

$$\begin{aligned}
& \beta_{0,d+2}(I^{\text{lex}}) \\
= & \dim_k(I^{\text{lex}})_{d+2} - \dim_k(R_1(I^{\text{lex}})_{d+1}) \\
(2.1) \quad = & [\dim_k R_{d+2} - \dim_k(R_1(I^{\text{lex}})_{d+1})] - [\dim_k R_{d+2} - \dim_k(I^{\text{lex}})_{d+2}] \\
= & \mathbf{H}_{R/I^{\text{lex}}}(d+1)^{\langle d+1 \rangle} - \mathbf{H}_{R/I^{\text{lex}}}(d+2) \\
= & \mathbf{H}_{R/I}(d+1)^{\langle d+1 \rangle} - \mathbf{H}_{R/I}(d+2) \quad (\because \mathbf{H}_{R/I}(t) = \mathbf{H}_{R/I^{\text{lex}}}(t) \text{ for every } t).
\end{aligned}$$

It follows from equation (2.1) that

$$\begin{aligned}
\beta_{0,d+2}(I) &= \dim_k(I_{d+2}) - \dim_k(\bar{I}_{d+2}) \\
&= [\dim_k R_{d+2} - \dim_k(\bar{I}_{d+2})] - [\dim_k R_{d+2} - \dim_k(I_{d+2})] \\
&= \mathbf{H}_{R/\bar{I}}(d+2) - \mathbf{H}_{R/I}(d+2) \\
&= (\mathbf{H}_{R/I}(d+1)^{\langle d+1 \rangle} - \mathbf{H}_{R/I}(d+2)) - (\mathbf{H}_{R/I}(d+1)^{\langle d+1 \rangle} - \mathbf{H}_{R/\bar{I}}(d+2)) \\
&= (\mathbf{H}_{R/I}(d+1)^{\langle d+1 \rangle} - \mathbf{H}_{R/I}(d+2)) - (\mathbf{H}_{R/\bar{I}}(d+1)^{\langle d+1 \rangle} - \mathbf{H}_{R/\bar{I}}(d+2)) \\
&\quad (\because \mathbf{H}_{R/I}(d+1) = \mathbf{H}_{R/\bar{I}}(d+1)) \\
&= \beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(\bar{I}^{\text{lex}}) \quad (\because \text{equation (2.1)}).
\end{aligned}$$

(c) Note that $\text{Gin}(I)_{d+1} = \text{Gin}(\bar{I})_{d+1}$. Hence we have

$$\begin{aligned}
& \beta_{0,d+2}(I) \\
= & \dim_k(I_{d+2}) - \dim_k(\bar{I}_{d+2}) \\
= & \dim_k(\text{Gin}(I)_{d+2}) - \dim_k(\text{Gin}(\bar{I})_{d+2}) \\
= & [\dim_k(\text{Gin}(I)_{d+2}) - \dim_k(R_1 \text{Gin}(I)_{d+1})] - [\dim_k(\text{Gin}(\bar{I})_{d+2}) - \dim_k(R_1 \text{Gin}(\bar{I})_{d+1})] \\
& (\because \text{Gin}(I)_{d+1} = \text{Gin}(\bar{I})_{d+1}) \\
= & \beta_{0,d+2}(\text{Gin}(I)) - \beta_{0,d+2}(\text{Gin}(\bar{I})),
\end{aligned}$$

which completes the proof. \square

Lemma 2.9. *Let $I \subset R = k[x_1, x_2, x_3]$ be a homogenous ideal and let that $A = R/I$ be a graded Artinian algebra. Then, for every $d > 0$,*

- (a) $\beta_{1,d}(I^{\text{lex}}) - \beta_{1,d}(I) = [\beta_{0,d}(I^{\text{lex}}) - \beta_{0,d}(I)] + [\beta_{2,d}(I^{\text{lex}}) - \beta_{2,d}(I)]$.
- (b) $\beta_{1,d}(\text{Gin}(I)) - \beta_{1,d}(I) = [\beta_{0,d}(\text{Gin}(I)) - \beta_{0,d}(I)] + [\beta_{2,d}(\text{Gin}(I)) - \beta_{2,d}(I)]$.

Proof. It is immediate by the cancellation principle. \square

3. AN h -VECTOR OF A GRADED ARTINIAN LEVEL ALGEBRA HAVING THE WLP

In this section, we think of h -vectors of a graded Artinian level algebra with the WLP and we shall prove that some of graded Artinian O-sequences are not level using generic initial ideals. Moreover, we assume that $R = k[x_1, \dots, x_n]$ is an n -variable polynomial ring over a field k with characteristic 0.

For positive integers h and i , h can be written uniquely in the form

$$h = h_{(i)} := \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \dots > m_j \geq j \geq 1$. This expansion for h is called the *i-binomial expansion* of h . For such h and i , we define

$$\begin{aligned}(h_{(i)})^- &:= \binom{m_i - 1}{i} + \binom{m_{i-1} - 1}{i-1} + \dots + \binom{m_j - 1}{j}, \\ (h_{(i)})_+^+ &:= \binom{m_i + 1}{i+1} + \binom{m_{i-1} + 1}{i} + \dots + \binom{m_j + 1}{j+1}.\end{aligned}$$

Let $\mathbf{H} = \{h_i\}_{i \geq 0}$ be the Hilbert function of a graded ring A . For simplicity in the notation we usually rewrite $((h_i)_{(i)})^-$ and $((h_i)_{(i)})_+^+$ as $(h_i)^-$ and $(h_i)_+^+$, respectively. Recall that we sometimes use another simpler notation $h^{\langle i \rangle}$ for $(h_i)_+^+$ and define $0^{\langle i \rangle} = 0$.

A well known result of Macaulay is the following theorem.

Theorem 3.1 (Macaulay). *Let $\mathbf{H} = \{h_i\}_{i \geq 0}$ be a sequence of non-negative integers such that $h_0 = 1$, $h_1 = n$, and $h_i = 0$ for every $i > e$. Then \mathbf{H} is the h-vector of some standard graded Artinian algebra if and only if, for every $1 \leq d \leq e - 1$,*

$$h_{d+1} \leq (h_d)_+^+ = h_d^{\langle d \rangle}.$$

We use a generic initial ideal with respect to the reverse lexicographic order to obtain results in Section 3. Note that, by Green's hyperplane restriction theorem (see [12]), we have that

$$(3.1) \quad \mathbf{H}(R/(J + x_n), d) \leq (\mathbf{H}(R/J, d))^-,$$

and the equality holds when J is a strongly stable ideal of R . In particular, the equality holds for any lex-segment ideal since a lex-segment ideal J of $R = k[x_1, \dots, x_n]$ is also a strongly stable ideal.

The following lemma will be used often in this section.

Lemma 3.2. *Let $A = R/I$ be an Artinian k -algebra and let L be a general linear form.*

(a) *If*

$$\dim_k(0 : L)_d > (n - 1) \dim_k(0 : L)_{d+1}$$

for some $d > 0$, then A has a socle element in degree d .

(b) *Let $h(A) = (h_0, h_1, \dots, h_s)$ be the h-vector of A . Then, we have*

$$(3.2) \quad h_d - h_{d+1} \leq \dim_k(0 : L)_d \leq h_d - h_{d+1} + (h_{d+1})^-.$$

In particular, $\dim_k(0 : L)_d = h_d - h_{d+1}$ if and only if $d \geq r_1(A)$.

Proof. (a) Consider a map $\varphi : (0 : L)_d \rightarrow \bigoplus^{n-1} (0 : L)_{d+1}$, defined by $\varphi(F) = (x_1 F, \dots, x_{n-1} F)$. Since L is a general linear form, we may assume that the kernel of this map is exactly $\text{soc}(A)_d$. Since $\dim_k(0 : L)_d > (n - 1) \dim_k(0 : L)_{d+1}$, the map φ is not injective and we obtain the desired result.

(b) Consider the following exact sequence

$$0 \rightarrow (0 : L)_d \rightarrow A_d \xrightarrow{\times L} A_{d+1} \rightarrow (A/LA)_{d+1} \rightarrow 0.$$

Then we have

$$(3.3) \quad \dim_k(0 : L)_d = h_d - h_{d+1} + \dim_k[A/(L)A]_{d+1},$$

and thus $h_d - h_{d+1} \leq \dim_k(0 : L)_d$. The right hand side of the inequality (3.2) follows from Green's hyperplane restriction theorem, i.e., $\dim_k[A/(L)A]_{d+1} \leq (h_{d+1})^-$.

Moreover, $\dim_k(0 : L)_d = h_d - h_{d+1}$ if and only if $\dim_k[A/(L)A]_{d+1} = 0$, and it is equivalent to $d \geq r_1(A)$ by the definition of $r_1(A)$. \square

Remark 3.3. Let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the h -vector of a graded Artinian level algebra $A = R/I$ and L is a general linear form of A . In general, it is not easy to find the reduction number $r_1(A)$ based on its h -vector. However if $h_{d+1} \leq d+1$ then $(h_{d+1})^- = 0$, and thus $\dim_k(0 : L)_d = h_d - h_{d+1}$. Hence $d \geq r_1(A)$ by Lemma 3.2. In other words,

$$r_1(A) \leq \min\{k \mid h_{k+1} \leq k+1\}.$$

Proposition 3.4. Let $R = k[x_1, \dots, x_n]$ and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the h -vector of a graded Artinian level algebra $A = R/I$ with socle degree s . Suppose that $h_{d-1} > h_d$ for some $d \geq r_1(A)$. Then

- (a) $h_{d-1} > h_d > \dots > h_{s-1} > h_s > 0$, and
- (b) $h_{t-1} - h_t \leq (n-1)(h_t - h_{t+1})$ for all $d \leq t \leq s$.

Proof. (a) First of all, note that, by Lemma 3.2 (b), $h_t - h_{t+1} = \dim_k(0 : L)_t$ for every $t \geq r_1(A)$. Hence we have that

$$h_{d-1} > h_d \geq h_{d+1} \geq \dots \geq h_s.$$

Now assume that there is $t \geq d$ such that $h_{t-1} > h_t = h_{t+1}$. Since $t \geq r_1(A)$, we know that, by Lemma 3.2 (b),

$$\dim_k(0 : L)_{t-1} \geq h_{t-1} - h_t > 0 \quad \text{and} \quad \dim_k(0 : L)_t = 0.$$

Hence there is a socle element of A in degree $t-1$, which is a contradiction since A is level. This means that $h_t > h_{t+1}$ for every $t \geq d-1$.

(b) Since A is a level algebra and $\dim_k(0 : L)_t = h_{t-1} - h_t$, the result follows directly from Lemma 3.2 (a). \square

Remark 3.5. Let I be a homogeneous ideal of $R = k[x_1, \dots, x_n]$ such that R/I has the WLP with a Lefschetz element L and let $\mathbf{H}(R/I, d-1) > \mathbf{H}(R/I, d)$ for some d . Now we consider the following exact sequence

$$(3.4) \quad (R/I)_{d-1} \xrightarrow{\times L} (R/I)_d \rightarrow (R/(I + (L)))_d \rightarrow 0.$$

Since R/I has the WLP and $\mathbf{H}(R/I, d-1) > \mathbf{H}(R/I, d)$, the above multiplication map cannot be injective, but surjective. In other words, $(R/(I + (L)))_d = 0$. This implies that $d > r_1(R/I)$ by Lemma 2.7.

The following theorem shows a useful condition to be a level O-sequence with the WLP.

Theorem 3.6. Let $R = k[x_1, \dots, x_n]$, $n \geq 3$ and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the Hilbert function of a graded Artinian level algebra $A = R/I$ having the WLP. Then,

- (a) the Hilbert function \mathbf{H} is a strictly unimodal O-sequence

$$h_0 < h_1 < \dots < h_{r_1(A)} = \dots = h_\theta > \dots > h_{s-1} > h_s$$

such that the positive part of the first difference $\Delta \mathbf{H}$ is an O-sequence, and

- (b) $h_{d-1} - h_d \leq (n-1)(h_d - h_{d+1})$ for $s \geq d > \theta$.

Proof. (a) First, note that, by Proposition 3.5 in [22], \mathbf{H} is a unimodal O-sequence such that the positive part of the first difference is an O-sequence. Hence it suffices to show that \mathbf{H} is strictly unimodal.

If $d \leq r_1(A)$, then $\mathbf{H}_{R/(I+L)}(d) \neq 0$ by the definition of $r_1(A)$, and so the multiplication map $\times L$ is not surjective in equation (3.4). In other words, the multiplication map $\times L$ is injective since A has the WLP. Thus, we have a short exact sequence as follows

$$0 \rightarrow (R/I)_{d-1} \xrightarrow{\times L} (R/I)_d \rightarrow (R/(I + (L)))_d \rightarrow 0.$$

Hence we obtain that

$$\begin{aligned}\mathbf{H}_A(d) &= \mathbf{H}_A(d-1) + \mathbf{H}_{R/(I+L)}(d) \\ &> \mathbf{H}_A(d-1) \quad (\because \mathbf{H}_{R/(I+L)}(d) \neq 0),\end{aligned}$$

and so the Hilbert function of A is strictly increasing up to $r_1(A)$.

Moreover, by Proposition 3.4 (a), \mathbf{H} is strictly decreasing in degrees $d \geq \theta$, where

$$\theta := \min\{t \mid h_t > h_{t+1}\}.$$

(b) The result follows directly from Proposition 3.4 (b). \square

Remark 3.7. Theorem 3.6 gives us a necessary condition when a numerical sequence becomes a level O-sequence with the WLP. In general, this condition is not sufficient. One can find many non-level sequences satisfying the inequality of Theorem 3.6 in [15].

In [15], they gave some of “non-level sequences” using homological method, which is the combinatorial notion of the cancellation of shifts in the minimal free resolutions of the lex-segment ideals associated to the given homogenous ideals.

In this section, we shall use generic initial ideals, instead of the lex-segment ideals. First note that, by Bigatti-Hulett-Pardue Theorem, the worst minimal free resolution of a homogenous ideal I depends on only the Hilbert function of I . Unfortunately, we cannot apply their theorem to obtain the minimal free resolutions of the generic initial ideals. However, we can find Betti-numbers $\beta_{i,d+i}(\text{Gin}(I))$ for $d > r_1(A)$ and $i \geq 0$, which depends on only the given Hilbert function (see Corollary 3.10).

For the rest of this section, we need the following useful results.

Lemma 3.8. Let J be a stable ideal of R and let T_1, \dots, T_r be the monomials which form a k -basis for $((J : x_n)/J)_{d-1}$ then

$$\{x_n T_1, \dots, x_n T_r\} = \{T \in \mathcal{G}(J)_d \mid x_n \text{ divides } T\}.$$

In particular,

$$\dim_k ((J : x_n)/J)_{d-1} = |\{T \in \mathcal{G}(J)_d \mid x_n \text{ divides } T\}|.$$

Proof. For every $T = x_n T' \in \mathcal{G}(J)_d$, we have that $x_n T' \in J_d \subset J$, i.e., $T' \in (J : x_n)_{d-1}$, and thus $\overline{T}' \in ((J : x_n)/J)_{d-1} = \langle \overline{T}_1, \dots, \overline{T}_r \rangle$. However, since T' and T_i are all monomials of $(J : x_n)_{d-1}$ in degree $d-1$, we have that $T' = T_i$ for some i , and hence $T = x_n T' \in \{x_n T_1, \dots, x_n T_r\}$.

Conversely, note that $T_i \notin J_{d-1}$ and $x_n T_i \in J_d$ for every $i = 1, \dots, r$. If $x_n T_i \notin \mathcal{G}(J)_d$ for some $i = 1, \dots, r$, then $x_n T_i \in R_1 J_{d-1}$. Since $T_i \notin J_{d-1}$, we see that

$$x_n T_i = x_j U$$

for some monomial $U \in J_{d-1}$ and $j < n$. Hence we have that

$$x_n \mid U.$$

Moreover, since J is a stable monomial ideal, for every $\ell < n$,

$$\frac{x_\ell}{x_n} U \in J_{d-1}.$$

In particular, we have

$$T_i = \frac{x_j}{x_n} U \in J_{d-1},$$

which is a contradiction. Therefore, $x_n T_i \in \mathcal{G}(J)_d$, for every $i = 1, \dots, r$, as we wished. \square

Using the previous lemma, we obtain the following proposition, which we know the difference between h_d and h_{d+1} when $d > r_1(A)$.

Proposition 3.9. Let $A = R/I$ be a graded Artinian algebra with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_s)$ and let $J = \text{Gin}(I)$. If $d \geq r_1(A)$ then,

$$|\{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\}| = h_d - h_{d+1}.$$

Moreover, if $d > r_1(A)$,

$$|\mathcal{G}(J)_{d+1}| = |\{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\}| = h_d - h_{d+1}.$$

Proof. Consider the following exact sequence:

$$0 \rightarrow ((J : x_n)/J)_d \rightarrow (R/J)_d \xrightarrow{\times x_n} (R/J)_{d+1} \rightarrow (R/J + (x_n))_{d+1} \rightarrow 0.$$

Note that $\mathbf{H}(R/I, t) = \mathbf{H}(R/J, t)$ for every $t \geq 0$. So

$$(3.5) \quad \begin{aligned} \dim_k((J : x_n)/J)_d + \dim_k(R/J)_{d+1} &= \dim_k(R/J)_d + \dim_k(R/J + (x_n))_{d+1}, \\ \Leftrightarrow \dim_k((J : x_n)/J)_d + h_{d+1} &= h_d + \dim_k(R/J + (x_n))_{d+1}. \end{aligned}$$

Moreover, by Theorem 2.1, Theorem 2.6, and Lemma 2.7, we have

$$\begin{aligned} r_1(R/I) &= r_1(R/J) \\ &= \min\{\ell \mid \mathbf{H}(R/J + (x_n), \ell + 1) = 0\}, \end{aligned}$$

which means $\mathbf{H}(R/J + (x_n), d + 1) = 0$ for every $d \geq r_1(R/I)$. Hence, from equation (3.5), we obtain

$$(3.6) \quad \dim_k((J : x_n)/J)_d = |\{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\}| = h_d - h_{d+1}.$$

Now suppose that $d > r_1(A)$. Then it is obvious that

$$(3.7) \quad \{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\} \subseteq \mathcal{G}(J)_{d+1}.$$

Conversely, note that $x_{n-1}^d \in J$ from the first equality of Lemma 2.7. Since J is a strongly stable ideal, J_d has to contain all monomials U of degree d such that

$$\text{supp}(U) := \{i \mid x_i \text{ divides } U\} \subseteq \{1, \dots, n-1\}.$$

This implies $\overline{\mathbf{m}}_d \subseteq J_d$ where $\overline{\mathbf{m}} = (x_1, \dots, x_{n-1})^d$. Thus we have

$$R_1 \overline{\mathbf{m}}_d \subseteq J_{d+1}.$$

Therefore, for every $T \in \mathcal{G}(J)_{d+1}$, we have $x_n \mid T$, and so

$$(3.8) \quad \mathcal{G}(J)_{d+1} \subseteq \{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\}.$$

It follows from equations (3.7) and (3.8) that

$$(3.9) \quad \mathcal{G}(J)_{d+1} = \{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\},$$

and hence

$$|\mathcal{G}(J)_{d+1}| = \dim_k((J : x_n)/J)_d = h_d - h_{d+1},$$

as we wished. \square

Corollary 3.10. Let $A = R/I$ be a graded Artinian algebra with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_s)$. If $d > r_1(A)$ then, for all $i \geq 0$,

$$\beta_{i, i+(d+1)}(\text{Gin}(I)) = (h_d - h_{d+1}) \binom{n-1}{i}.$$

Proof. By Proposition 3.9,

$$|\mathcal{G}(\text{Gin}(I))_{d+1}| = |\{T \in \mathcal{G}(\text{Gin}(I))_{d+1} \mid x_n \text{ divides } T\}| = h_d - h_{d+1}$$

for every $d > r_1(A)$, and thus the result follows from Theorem 2.4. \square

Recall that a homogeneous ideal I is *m-regular* if, in the minimal free resolution of I , for all $p \geq 0$, every p -th syzygy has degree $\leq m + p$. The regularity of I , $\text{reg}(I)$, is the smallest such m .

In [2] and [19], they proved that the regularity of $\text{Gin}(I)$ is the largest degree of a generator of $\text{Gin}(I)$. Moreover, Bayer and Stillman [2] showed the regularity of I is equal to the regularity of $\text{Gin}(I)$.

Theorem 3.11 ([2], [19]). *For any homogeneous ideal I , using the reverse lexicographic order,*

$$\text{reg}(I) = \text{reg}(\text{Gin}(I)).$$

Theorem 3.12 (Crystallization Principle, [1] and [19]). *Let I be a homogeneous ideal generated in degrees $\leq d$. Assume that there is a monomial order τ such that $\text{Gin}_\tau(I)$ has no generator in degree $d + 1$. Then $\text{Gin}_\tau(I)$ is generated in degrees $\leq d$ and I is d -regular.*

Lemma 3.13. *Let $R = k[x_1, x_2, x_3]$ and let $A = R/I$ be an Artinian algebra and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the Hilbert function of $A = R/I$. Suppose that, for $t > 0$,*

- (a) $\text{soc}(A)_{t-2} = 0$,
- (b) $\beta_{1,t+1}(\text{Gin}(I)) = \beta_{2,t+1}(\text{Gin}(I))$.

Then $(I_{\leq t})$ is t -regular and

$$(3.10) \quad h_{t-1} - h_t \leq \dim_k \text{soc}(A)_{t-1} \leq h_{t-1} - h_t + (h_t)^-$$

In particular, if $t > r_1(A)$ then

$$\dim_k (\text{soc}(A)_{t-1}) = h_{t-1} - h_t.$$

Proof. Let $\bar{I} = (I_{\leq t})$. Note that $\beta_{i,t+1}(\text{Gin}(I)) = \beta_{i,t+1}(\text{Gin}(\bar{I}))$ for $i = 1, 2$ and $\beta_{0,t+1}(\bar{I}) = 0$. Furthermore, since I and \bar{I} agree in degree $\leq t$ and $\text{soc}(A)_{t-2} = 0$, we see that $\beta_{2,t+1}(I) = \beta_{2,t+1}(\bar{I}) = 0$.

Applying Lemma 2.9 (b) the ideal \bar{I} , we have that

$$\begin{aligned} \beta_{1,t+1}(\text{Gin}(\bar{I})) - \beta_{1,t+1}(\bar{I}) &= (\beta_{0,t+1}(\text{Gin}(\bar{I})) - \beta_{0,t+1}(\bar{I})) + (\beta_{2,t+1}(\text{Gin}(\bar{I})) - \beta_{2,t+1}(\bar{I})) \\ \Rightarrow -\beta_{1,t+1}(\bar{I}) &= (\beta_{0,t+1}(\text{Gin}(\bar{I})) - \beta_{0,t+1}(\bar{I})) - \beta_{2,t+1}(\bar{I}) \quad (\because \beta_{1,t+1}(\text{Gin}(\bar{I})) = \beta_{2,t+1}(\text{Gin}(\bar{I}))) \\ \Rightarrow -\beta_{1,t+1}(\bar{I}) &= \beta_{0,t+1}(\text{Gin}(\bar{I})) \quad (\because \beta_{0,t+1}(\bar{I}) = \beta_{2,t+1}(\bar{I}) = 0) \\ \Rightarrow \beta_{0,t+1}(\text{Gin}(\bar{I})) &= 0. \end{aligned}$$

Thus, by Theorem 3.12, the ideal $\bar{I} = (I_{\leq t})$ is t -regular.

Let $\bar{A} = R/\bar{I}$. For a general linear form L , consider the following exact sequence

$$(3.11) \quad 0 \rightarrow (0 :_{\bar{A}} L)_{t-1} \rightarrow (R/\bar{I})_{t-1} \xrightarrow{\times L} (R/\bar{I})_t \rightarrow (R/\bar{I} + (L))_t \rightarrow 0.$$

After we replace \bar{I} and \bar{A} by $\text{Gin}(\bar{I})$ and $\tilde{A} = R/\text{Gin}(\bar{I})$, respectively, we can rewrite equation (3.11) as

$$(3.12) \quad 0 \rightarrow (0 :_{\tilde{A}} x_3)_{t-1} \rightarrow (R/\text{Gin}(\bar{I}))_{t-1} \xrightarrow{\times x_3} (R/\text{Gin}(\bar{I}))_t \rightarrow (R/\text{Gin}(\bar{I}) + (x_3))_t \rightarrow 0.$$

Then, by Theorem 2.1, we know that

$$\begin{aligned} \dim_k (0 :_{\tilde{A}} x_3)_{t-1} &= \dim_k ((\text{Gin}(\bar{I}) : x_3)/\text{Gin}(\bar{I}))_{t-1} \\ &= h_{t-1} - h_t + \dim_k (R/\text{Gin}(\bar{I}) + (x_3))_t \\ &= h_{t-1} - h_t + \dim_k (R/\bar{I} + (L))_t \\ &= \dim_k (0 :_{\bar{A}} L)_{t-1}. \end{aligned}$$

On the other hand, by Lemma 3.8,

$$\begin{aligned}\dim_k((\text{Gin}(\bar{I}) : x_3)/\text{Gin}(\bar{I}))_{t-1} &= |\{T \in \mathcal{G}(\text{Gin}(\bar{I}))_t \mid x_3 \text{ divides } T\}| \\ &= \beta_{2,t+2}(\text{Gin}(\bar{I})),\end{aligned}$$

and by Lemma 3.2 (b)

$$(3.13) \quad h_{t-1} - h_t \leq \dim_k((0 :_{\bar{A}} L)_{t-1}) \leq h_{t-1} - h_t + (h_t)^-.$$

Note that, by Theorem 3.12, $\beta_{1,t+2}(\text{Gin}(\bar{I})) = 0$ since $\bar{I} = (I_{\leq t})$ is t -regular. Moreover, since I and \bar{I} agree in degree $\leq t$, we have $\beta_{2,t+2}(I) = \beta_{2,t+2}(\bar{I})$. Hence, by Theorem 2.2,

$$\begin{aligned}(3.14) \quad \dim_k \text{soc}(A)_{t-1} &= \beta_{2,t+2}(I) \\ &= \beta_{2,t+2}(\bar{I}) \\ &= \beta_{2,t+2}(\text{Gin}(\bar{I})) \quad (\because \beta_{1,t+2}(\text{Gin}(\bar{I})) = 0) \\ &= \dim_k(0 :_{\bar{A}} L)_{t-1}.\end{aligned}$$

Hence it follows from equations (3.13) and (3.14), we obtain the inequality (3.10). Moreover, by Lemma 3.2 (b), we have

$$\dim_k(\text{soc}(A)_{t-1}) = h_{t-1} - h_t \quad \text{for } t > r_1(A),$$

as we wished. \square

Theorem 3.14. *Let $A = R/I$ be an Artinian algebra of codimension 3 with socle degree s . If*

$$(3.15) \quad \beta_{1,d+2}(\text{Gin}(I)) = \beta_{2,d+2}(\text{Gin}(I)) > 0.$$

for some $d < s$, then A is not level.

Proof. Assume A is level. Then $\beta_{2,d+2}(I) = \text{soc}(A)_{d-1} = 0$, and hence, by Lemma 3.13, $\bar{I} = (I_{\leq d+1})$ is $(d+1)$ -regular.

Let $\bar{A} = R/\bar{I}$. Note that $\text{soc}(A)_d = \text{soc}(\bar{A})_d$ since A and \bar{A} agree in degree $\leq d+1$, i.e.,

$$\dim_k \text{soc}(A)_d = \beta_{2,d+3}(I) = \beta_{2,d+3}(\bar{I}) = \dim_k \text{soc}(\bar{A})_d.$$

For a general linear form L , by Lemmas 3.2 (a) and 3.8, we have that

$$\begin{aligned}0 &< \beta_{2,d+2}(\text{Gin}(I)) && (\because \text{by assumption}) \\ &= \sum_{T \in \mathcal{G}(\text{Gin}(I))_d} \binom{m(T)-1}{2} \\ &= \dim_k[(\text{Gin}(I) : x_3)/\text{Gin}(I)]_{d-1} && (\because \text{by Lemma 3.8}) \\ &= \dim_k[(I : L)/I]_{d-1} \\ &\leq 2 \dim_k[(I : L)/I]_d && (\because \text{by Lemma 3.2 (a) and } \text{soc}(A)_{d-1} = 0).\end{aligned}$$

Note that, in the similar way, we have $\beta_{2,d+3}(\text{Gin}(I)) = \dim_k[(I : L)/I]_d$. Hence

$$\beta_{2,d+3}(\text{Gin}(I)) > 0.$$

Since $\bar{I} = (I_{\leq d+1})$ is $(d+1)$ -regular and $\text{reg}(\bar{I}) = \text{reg}(\text{Gin}(\bar{I}))$ by Theorem 3.11, we have that

$$\begin{aligned}\beta_{0,d+3}(\text{Gin}(\bar{I})) &= \beta_{1,d+3}(\text{Gin}(\bar{I})) = 0, \\ \beta_{0,d+3}(\bar{I}) &= \beta_{1,d+3}(\bar{I}) = 0.\end{aligned}$$

Thus, by Lemma 2.9 (b),

$$\beta_{2,d+3}(\bar{I}) = \beta_{2,d+3}(\text{Gin}(\bar{I})) > 0,$$

which follows that R/\bar{I} has a socle element in degree d , so does R/I . This is a contradiction, and thus we complete the proof. \square

Remark 3.15. Now we shall show that there is a level O-sequence satisfying Theorem 3.6 (a) and (b), but it cannot be the Hilbert function of an Artinian algebra with the WLP.

Consider an h -vector $\mathbf{H} = (1, 3, 6, 10, 8, 7)$, which was given in [15]. Furthermore, it has been shown that there is a level algebra of codimension 3 with Hilbert function \mathbf{H} in [15]. They also raised a question if there exists a codimension 3 graded level algebra having the WLP with Hilbert function \mathbf{H} . Note that this is a codimension 3 level O-sequence which satisfies the condition in Theorem 3.6.

Now suppose that there is an Artinian level algebra $A = R/I$ having the WLP with Hilbert function \mathbf{H} . In [15], they gave several results about level or non-level sequences of graded Artinian algebras. One of the tools they used was the fact that Betti numbers of a homogeneous ideal I can be obtained by cancellation of the Betti numbers of I^{lex} . However, in this case, it is not available if \mathbf{H} can be the Hilbert function of an Artinian level algebra having the WLP based on the Betti numbers of I^{lex} .

In fact, the Betti diagram of R/I^{lex} is

| total: | 1 | - | - | - |
|--------|---|---|---|---|
| <hr/> | | | | |
| 0: | 1 | - | - | - |
| 1: | 0 | 0 | 0 | 0 |
| 2: | 0 | 0 | 0 | 0 |
| 3: | 0 | 7 | 9 | 3 |
| 4: | 0 | 2 | 4 | 2 |
| <hr/> | | | | |
| | | | | |

and thus we cannot decide if there is a socle element of R/I in degree 3.

Note that, by Theorem 3.6, $r_1(A) = 3$ since A has the WLP. Hence, by Corollary 3.10,

$$\begin{aligned}\beta_{2,6}(\text{Gin}(I)) &= (h_4 - h_5)\binom{2}{2} = 2 \cdot 1 = 2, \quad \text{and} \\ \beta_{1,6}(\text{Gin}(I)) &= (h_5 - h_6)\binom{2}{1} = 1 \cdot 2 = 2.\end{aligned}$$

Therefore, by Theorem 3.14, there is a socle element in A in degree 3, which is a contradiction. In other words, any Artinian level algebra A with Hilbert function \mathbf{H} does not have the WLP.

Remark 3.16. In general, Theorem 3.14 is not true if equation (3.15) holds in the socle degree. For example, we consider a Gorenstein sequence

| d | 0 | 1 | 2 | 3 | 4 |
|-------|---|---|---|---|---|
| h_d | 1 | 3 | 6 | 3 | 1 |

By Remark 3.3, $r_1(A) \leq 2$. Hence

$$\begin{aligned}\beta_{1,6}(\text{Gin}(I)) &= (h_4 - h_5)\binom{2}{1} = 1 \cdot 2 = 2, \quad \text{and} \\ \beta_{2,6}(\text{Gin}(I)) &= (h_3 - h_4)\binom{2}{2} = 2 \cdot 1 = 2.\end{aligned}$$

Note this satisfies the condition of Theorem 3.14 in the socle degree, but it is a level sequence.

Remark 3.17. Let $A = R/I$ be an Artinian algebra and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the Hilbert function of $A = R/I$. Then an ideal $(I_{\leq d+1})$ is $(d+1)$ -regular, if the Hilbert function \mathbf{H} of A has the maximal growth in degree $d > 0$, i.e. $h_{d+1} = h_d^{\langle d \rangle}$. In particular, if $h_d = h_{d+1} = \ell \leq d$, then we know that $(I_{\leq d+1})$ is $(d+1)$ -regular. Recently, this result was improved in [1], that is, $(I_{\leq d+1})$ is $(d+1)$ -regular if $h_d = h_{d+1}$ and $r_1(A) < d$.

Note that, by Lemma 3.2, the k -vector space dimension of $(0 : L)_d$ in degree $d \geq r_1(A)$ is $h_d - h_{d+1}$. By Proposition 3.4, we have a bound for the growth of Hilbert function of $(0 : L)$

in degree $d \geq r_1(A)$ if an Artinian algebra A has no socle elements in degree d . Theorem 3.19 shows that a similar result still holds on the maximal growth of the Hilbert function of $(0 : L)$ in codimension three case.

Lemma 3.18. *Let $R = k[x_1, \dots, x_n]$ and let $A = R/I$ be an Artinian algebra with an h -vector $\mathbf{H} = (1, 3, h_2, \dots, h_s)$. If $h_{d-1} - h_d = (n-1)(h_d - h_{d+1})$ for $r_1(A) < d < s$, then*

$$\beta_{(n-1), (n-1)+d}(\text{Gin}(I)) = \beta_{(n-2), (n-1)+d}(\text{Gin}(I)).$$

Proof. Let $J = \text{Gin}(I)$. By Proposition 3.9, we have that

$$\begin{aligned} \beta_{(n-1), (n-1)+d}(J) &= \sum_{T \in \mathcal{G}(J)_d} \binom{m(T) - 1}{n-1} \\ &= h_{d-1} - h_d. \end{aligned}$$

Moreover, by Corollary 3.10,

$$\begin{aligned} \beta_{(n-2), (n-1)+d}(J) &= \beta_{(n-2), (n-2)+(d+1)}(J) \\ &= (h_d - h_{d+1}) \binom{n-1}{n-2} \\ &= (n-1)(h_d - h_{d+1}) \\ &= h_{d-1} - h_d \quad (\because \text{by given condition}) \\ &= \beta_{(n-1), (n-1)+d}(J), \end{aligned}$$

as we wished. \square

Theorem 3.19. *Let $R = k[x_1, x_2, x_3]$ and let $A = R/I$ be an Artinian algebra with an h -vector $\mathbf{H} = (1, 3, h_2, \dots, h_s)$. If $\text{soc}(A)_{d-1} = 0$ and the Hilbert function of $(0 : L)$ has a maximal growth in degree d for $r_1(A) < d < s$, i.e., $h_{d-1} - h_d = 2(h_d - h_{d+1})$, then*

- (a) $(I_{\leq d+1})$ is $(d+1)$ -regular, and
- (b) $\dim_k \text{soc}(A)_d = h_d - h_{d+1}$.

Proof. By Lemma 3.18, we have

$$(3.16) \quad \beta_{1, d+2}(\text{Gin}(I)) = \beta_{2, d+2}(\text{Gin}(I)),$$

for $r_1(A) < d < s$, and the result immediately follows from Lemma 3.13. \square

Corollary 3.20. *Let $R = k[x_1, x_2, x_3]$ and let $A = R/I$ be an Artinian algebra with an h -vector $\mathbf{H} = (1, 3, h_2, \dots, h_s)$. If $h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0$ for $r_1(A) < d < s$, then A is not level.*

Proof. By Lemma 3.18, we have

$$\beta_{2, d+2}(\text{Gin}(I)) = \beta_{1, d+2}(\text{Gin}(I)) > 0,$$

and hence, by Theorem 3.14, A cannot be level, as we wanted. \square

Remark 3.21. Remark 3.16 shows Corollary 3.20 is not true if $d = s$. However we know $h_{s-1} \leq 3h_s$ by Theorem 3.6.

Example 3.22. Let $A = R/I$ be a codimension 3 Artinian algebra and let $r_1(A) < d < s$. If A has the Hilbert function

| | | | | | |
|-------|-----|--------|-------|-------|-----|
| d | ... | $d-1$ | d | $d+1$ | ... |
| h_d | ... | $a+3k$ | $a+k$ | a | ... |

such that $a > 0$ and $k > 0$, then by Corollary 3.20 A cannot be level since

$$h_{d-1} - h_d = 2k = 2(h_d - h_{d+1}) \Leftrightarrow \beta_{2, d+2}(\text{Gin}(I)) = \beta_{1, d+2}(\text{Gin}(I)) > 0.$$

For the codimension 3 case, we have the following theorem, which follows from Theorems 3.6 and 3.19 and Corollary 3.20, and so we shall omit the proof here.

Theorem 3.23. *Let $A = R/I$ be a graded Artinian level algebra of codimension 3 with the WLP and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the Hilbert function of A . Then,*

- (a) *the Hilbert function \mathbf{H} is a strictly unimodal O-sequence*

$$h_0 < h_1 < \dots < h_{r_1(A)} = \dots = h_\theta > \dots > h_{s-1} > h_s$$

such that the positive part of the first difference $\Delta\mathbf{H}$ is an O-sequence, and

- (b) $h_{d-1} - h_d < 2(h_d - h_{d+1})$ for $s > d > \theta$.
- (c) $h_{s-1} \leq 3h_s$.

One may ask if the converse of Theorem 3.23 holds. Before the end of this section, we give the following Question.

Question 3.24. Suppose that $\mathbf{H} = (1, 3, h_2, \dots, h_s)$ is the h -vector of a level algebra $A = R/I$ where $R = k[x_1, x_2, x_3]$. Is there a level algebra A with the WLP such that \mathbf{H} is the Hilbert function of A if $\mathbf{H} = (1, 3, h_2, \dots, h_s)$ satisfies the condition (a), (b) and (c) in Theorem 3.23?

4. THE LEX-SEGMENT IDEALS AND GRADED NON-LEVEL ARTINIAN ALGEBRAS

In this section, we shall find an answer to Question 1.1.

Theorem 4.1. *Let $R = k[x_1, x_2, x_3]$ and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the h -vector of a graded Artinian algebra $A = R/I$ with socle degree s . If*

$$h_{d-1} > h_d \quad \text{and} \quad h_d = h_{d+1} \leq 2d + 3,$$

then \mathbf{H} is not level.

Before we prove this theorem, we consider the following lemmas and the theorems.

Lemma 4.2. *Let J be a lex-segment ideal in $R = k[x_1, x_2, x_3]$ such that*

$$\mathbf{H}(R/J, i) = h_i$$

for every $i \geq 0$. Then

$$(4.1) \quad \dim_k ((J : x_3)/J)_i = h_i - h_{i+1} + (h_{i+1})^-$$

for such an i .

Proof. First of all, we consider the following exact sequence:

$$(4.2) \quad 0 \rightarrow ((J : x_3)/J)_i \rightarrow (R/J)_i \xrightarrow{\times x_3} (R/J)_{i+1} \rightarrow R/(J + (x_3))_{i+1} \rightarrow 0.$$

Using equations (3.1) and (4.2), we see that

$$(4.3) \quad \dim_k ((J : x_3)/J)_i = h_i - h_{i+1} + (h_{i+1})^-$$

for every $i \geq 0$ as we desired. \square

Since the following lemma is obtained easily from the property of the lex-segment ideal, we shall omit the proof here.

Lemma 4.3. Let I be the lex-segment ideal in $R = k[x_1, x_2, x_3]$ with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_s)$ where $h_d = d + i$ and $1 \leq i \leq \frac{d^2+d}{2}$. Then the last monomial of I_d is

$$\begin{aligned} & x_1 x_2^{i-1} x_3^{d-i}, & \text{for } 1 \leq i \leq d, \\ & x_1^2 x_2^{i-(d+1)} x_3^{(2d-1)-i}, & \text{for } d+1 \leq i \leq 2d-1, \\ & \vdots \\ & x_1^{d-1} x_2^{i-\frac{d^2+d-4}{2}} x_3^{\frac{d^2+d-2}{2}-i}, & \text{for } \frac{d^2+d-4}{2} \leq i \leq \frac{d^2+d-2}{2}, \\ & x_1^d, & \text{for } i = \frac{d^2+d}{2}. \end{aligned}$$

Theorem 4.4. Let $R = k[x_1, x_2, x_3]$ and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the h -vector of an Artinian algebra with socle degree s and

$$h_d = h_{d+1} = d + i, \quad h_{d-1} > h_d, \quad \text{and} \quad j := h_{d-1} - h_d$$

for $i = 1, 2, \dots, \frac{d^2+d}{2}$. Then, for every $1 \leq k \leq d$ and $1 \leq \ell \leq d$,

$$\beta_{1,d+2} = \begin{cases} 2k-1, & \text{for } (k-1)d - \frac{k(k-3)}{2} \leq i \leq (k-1)d - \frac{k(k-3)}{2} + (k-1), \\ 2k, & \text{for } (k-1)d - \frac{k(k-3)}{2} + k \leq i \leq kd - \frac{(k-1)k}{2}. \end{cases}$$

$$\beta_{2,d+2} = j + \ell, \quad \text{for } (\ell-1)d - \frac{(\ell-2)(\ell-1)}{2} < i \leq \ell d - \frac{(\ell-1)\ell}{2}.$$

Proof. Since $h_d = d + i$, the monomials not in I_d are the last $d + i$ monomials of R_d . By Lemma 4.3, the last monomial of $R_1 I_d$ is

$$\begin{aligned} & x_1 x_2^{i-1} x_3^{d-i+1}, & \text{for } i = 1, \dots, d, \\ & x_1^2 x_2^{i-(d+1)} x_3^{2d-i}, & \text{for } i = d+1, \dots, 2d-1, \\ & \vdots \\ & x_1^{d-1} x_2^{i-\frac{d^2+d-4}{2}} x_3^{\frac{d^2+d-2}{2}-i}, & \text{for } i = \frac{d^2+d-4}{2}, \frac{d^2+d-2}{2}, \\ & x_1^d x_3, & \text{for } i = \frac{d^2+d}{2}. \end{aligned}$$

In what follows, the first monomial of $I_{d+1} - R_1 I_d$ is

$$(4.4) \quad \begin{aligned} & x_2^{d+1}, & \text{for } i = 1, \\ & x_1 x_2^{i-2} x_3^{(d+2)-i}, & \text{for } i = 2, \dots, d, \\ & \vdots \\ & x_1^{d-1} x_2 x_3, & \text{for } i = \frac{d^2+d-2}{2}, \\ & x_1^{d-1} x_2^2, & \text{for } i = \frac{d^2+d}{2}. \end{aligned}$$

Note that

$$(4.5) \quad (d+i)^{\langle d \rangle} = (d+i) + k, \quad \text{for } i = (k-1)d - \frac{k(k-3)}{2}, \dots, kd - \frac{k(k-1)}{2}, \quad \text{and } k = 1, \dots, d.$$

We now calculate the Betti number

$$\beta_{1,d+2} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T) - 1}{1}.$$

Based on equation (4.4), we shall find this Betti number of each two cases for i as follows.

Case 1-1. $i = (k-1)d - \frac{k(k-3)}{2}$ and $k = 1, 2, \dots, d$.

By equation (4.5), I_{d+1} has k -generators, which are

$$x_1^{k-1} x_2^{(d+2)-k}, x_1^{k-1} x_2^{(d+1)-k} x_3, \dots, x_1^{k-1} x_2^{(d+3)-2k} x_3^{k-1}.$$

By the similar argument, I_{d+1} has k -generators including the element $x_1^{k-1}x_2^{(d+2)-k}$ for $i = (k-1)d - \frac{k(k-3)}{2} + 1, \dots, (k-1)d - \frac{k(k-3)}{2} + (k-1)$. Hence we have that

$$\beta_{1,d+2} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T) - 1}{1} = 2 \times (k-1) + 1 = 2k - 1.$$

Case 1-2. $i = (k-1)d - \frac{k(k-3)}{2} + k = (k-1)d - \frac{k(k-5)}{2}, \dots, kd - \frac{k(k-1)}{2}$ and $k = 1, 2, \dots, d$.

By equation (4.5), I_{d+1} has k -generators, which are

$$x_1^k x_2^{i - \left((k-1)d - \frac{k^2 - 3k - 2}{2} \right)} x_3^{kd - \frac{k^2 - k - 4}{2} - i}, \dots, x_1^k x_2^{i - \left((k-1)d - \frac{k(k-5)}{2} \right)} x_3^{\left(kd - \frac{k(k-3)}{2} + 1 \right) - i}.$$

Hence we have that

$$\beta_{1,d+2} = \sum_{T \in \mathcal{G}(I)_{d+1}} \binom{m(T) - 1}{1} = 2 \times k = 2k.$$

Now we move on to the Betti number:

$$\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T) - 1}{2}.$$

Recall $h_d = d + i$ and $j := h_{d-1} - h_d$. The computation of the Betti number of this case is much more complicated, and thus we shall find the Betti number of each four cases based on i and j .

Case 2-1. $(\ell-1)d - \frac{(\ell-2)(\ell-1)}{2} < i < \ell d - \frac{(\ell-1)\ell}{2}$ and $\ell = 1, 2, \dots, d$.

The last monomial of I_d for this case is

$$x_1^\ell x_2^{i - (\ell-1)d + \frac{\ell(\ell-3)}{2}} x_3^{\ell d - \frac{(\ell-1)\ell}{2} - i}.$$

Case 2-1-1. $(k-1)d - \frac{(k-1)k}{2} < i + j < kd - \frac{k(k+1)}{2}$ and $k = \ell, \ell + 1, \dots, d$.

Since first monomial of $I_d - R_1 I_{d-1}$ is

$$x_1^k x_2^{(i+j) - \left((k-1)d - \frac{(k-2)(k+1)}{2} \right)} x_3^{\left(kd - \frac{(k-1)(k+2)}{2} \right) - (i+j)},$$

we have $(j+k)$ -generators in I_d as follows:

$$\begin{aligned} & x_1^k x_2^{(i+j) - \left((k-1)d - \frac{(k-2)(k+1)}{2} \right)} x_3^{\left(kd - \frac{(k-1)(k+2)}{2} \right) - (i+j)}, \dots, x_1^k x_3^{d-k}, \\ & x_1^{(k-1)} x_2^{d-(k-1)}, x_1^{(k-1)} x_2^{(d-1)-(k-1)} x_3, \dots, x_1^{(k-1)} x_3^{d-(k-1)}, \\ & \vdots \\ & x_1^{\ell+1} x_2^{(d-1)-\ell}, x_1^{\ell+1} x_2^{(d-2)-\ell} x_3, \dots, x_1^{\ell+1} x_3^{(d-1)-\ell} \\ & x_1^\ell x_2^{d-\ell}, \dots, x_1^\ell x_2^{i - (\ell-1)d + \frac{\ell(\ell-3)}{2}} x_3^{\ell d - \frac{(\ell-1)\ell}{2} - i} \end{aligned}$$

and thus

$$\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T) - 1}{2} = j + \ell.$$

Case 2-1-2. $i + j = (k-1)d - \frac{(k-1)k}{2}$ and $k = \ell + 1, \dots, d$.

The first monomial of $I_d - R_1 I_{d-1}$ is

$$x_1^{k-1} x_2^{d-(k-1)},$$

and hence we have $(j+k)$ -generators in I_d as follows:

$$\begin{aligned} & x_1^{k-1}x_2^{d-(k-1)}, x_1^{k-1}x_2^{(d-1)-(k-1)}x_3, \dots, x_1^{k-1}x_3^{d-(k-1)}, \\ & \quad \vdots \\ & x_1^{\ell+1}x_2^{(d-1)-\ell}, x_1^{\ell+1}x_2^{(d-2)-\ell}x_3, \dots, x_1^{\ell+1}x_3^{(d-1)-\ell} \\ & x_1^\ell x_2^{d-\ell}, \dots, x_1^\ell x_2^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}}x_3^{\ell d-\frac{(\ell-1)\ell}{2}-i} \end{aligned}$$

and thus

$$\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T) - 1}{2} = j + \ell.$$

Case 2-2. $i = \ell d - \frac{(\ell-1)\ell}{2}$ and $\ell = 1, 2, \dots, d$.

The last monomial of I_d is

$$x_1^\ell x_2^{d-\ell}.$$

Case 2-2-1. $(k-1)d - \frac{(k-1)k}{2} < i+j < kd - \frac{k(k+1)}{2}$ and $k = \ell + 1, \dots, d$.

Since the first monomial of $I_d - R_1 I_{d-1}$ is

$$x_1^k x_2^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_3^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)},$$

we have $(j+k)$ -generators in I_d as follows:

$$\begin{aligned} & x_1^k x_2^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_3^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)}, \dots, x_1^k x_3^{d-k}, \\ & x_1^{(k-1)} x_2^{d-(k-1)}, x_1^{(k-1)} x_2^{(d-1)-(k-1)} x_3, \dots, x_1^{(k-1)} x_3^{d-(k-1)}, \\ & \quad \vdots \\ & x_1^{\ell+1} x_2^{(d-1)-\ell}, x_1^{\ell+1} x_2^{(d-2)-\ell} x_3, \dots, x_1^{\ell+1} x_3^{(d-1)-\ell} \\ & x_1^\ell x_2^{d-\ell}, \end{aligned}$$

and thus

$$\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T) - 1}{2} = j + \ell.$$

Case 2-2-2. $i+j = (k-1)d - \frac{(k-1)k}{2}$ and $k = \ell + 1, \dots, d$.

The first monomial of $I_d - R_1 I_{d-1}$ is

$$x_1^{(k-1)} x_2^{d-(k-1)},$$

and hence we have $(j+k)$ -generators in I_d as follows:

$$\begin{aligned} & x_1^{(k-1)} x_2^{d-(k-1)}, x_1^{(k-1)} x_2^{(d-1)-(k-1)} x_3, \dots, x_1^{(k-1)} x_3^{d-(k-1)}, \\ & \quad \vdots \\ & x_1^{\ell+1} x_2^{(d-1)-\ell}, x_1^{\ell+1} x_2^{(d-2)-\ell} x_3, \dots, x_1^{\ell+1} x_3^{(d-1)-\ell} \\ & x_1^\ell x_2^{d-\ell}, \end{aligned}$$

and thus

$$\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T) - 1}{2} = j + \ell,$$

as we wished. \square

Theorem 4.5. Let \mathbf{H} be as in equation (1.1) and $A = R/I$ be an algebra with Hilbert function \mathbf{H} such that $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$ for some $d < s$. Then A is **not** level.

Proof. Let L be a general linear form of A . By Lemma 3.2 (b), note that if $d \geq r_1(A)$, then

$$\dim_k(0 : L)_{d-1} \geq h_{d-1} - h_d > 0 \quad \text{and} \quad \dim_k(0 : L)_d = h_d - h_{d+1} = 0,$$

and thus, by Lemma 3.2 (a), R/I is not level. Hence we assume that $d < r_1(A)$ and A is a graded level algebra having Hilbert function \mathbf{H} . Let $\bar{I} = (I_{\leq d+1})$.

Claim. $\beta_{1,d+3}(\text{Gin}(\bar{I})) = 0$ and $\beta_{2,d+3}(\text{Gin}(\bar{I})) > 0$.

Proof of Claim. First we shall show that $\beta_{1,d+3}(\text{Gin}(\bar{I})) = 0$. By Lemma 2.9 (a),

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}),$$

and we have that

$$(4.6) \quad \begin{aligned} \beta_{1,d+2}(I^{\text{lex}}) - \beta_{1,d+2}(I) &= [\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)] + [\beta_{2,d+2}(I^{\text{lex}}) - \beta_{2,d+2}(I)] \\ \Rightarrow -\beta_{1,d+2}(I) &= [\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)] - \beta_{2,d+2}(I). \end{aligned}$$

Moreover, since $A = R/I$ is level, we know that $\beta_{2,d+2}(I) = 0$, and hence rewrite equation (4.6) as

$$0 \leq [\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)] = -\beta_{1,d+2}(I) \leq 0,$$

which follows from Lemma 2.8 (b) that

$$\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I) = \beta_{0,d+2}(\bar{I}^{\text{lex}}) = 0.$$

Also, by Lemma 2.8 (a), we have

$$\beta_{0,d+2}(\text{Gin}(\bar{I})) \leq \beta_{0,d+2}(\bar{I}^{\text{lex}}) = 0, \quad \text{i.e.,} \quad \beta_{0,d+2}(\text{Gin}(\bar{I})) = 0.$$

Since $\text{Gin}(\bar{I})$ is a Borel fixed monomial ideal, by Theorem 2.4,

$$\beta_{1,d+3}(\text{Gin}(\bar{I})) = 0.$$

Now we shall prove that $\beta_{2,d+3}(\text{Gin}(\bar{I})) > 0$. Let $J = \text{Gin}(\bar{I})$. Consider the following exact sequence

$$0 \rightarrow ((J : x_3)/J)_d \rightarrow (R/J)_d \xrightarrow{\times x_3} (R/J)_{d+1} \rightarrow (R/J + (x_3))_{d+1} \rightarrow 0.$$

Since $d < r_1(A)$, we know that

$$\begin{aligned} \dim_k((J : x_3)/J)_d &= h_d - h_{d+1} + \dim_k((R/J + (x_3))_{d+1}) \\ &= \dim_k((R/J + (x_n))_{d+1}) \quad (\because h_d = h_{d+1}) \\ &\neq 0. \end{aligned}$$

By Lemma 3.8,

$$\mathcal{G}(J)_{d+1} = \mathcal{G}(\text{Gin}(\bar{I}))_{d+1} \neq \emptyset,$$

and so there is a monomial $T \in \mathcal{G}(\text{Gin}(\bar{I}))_{d+1}$ such that $x_3 | T$. In other words,

$$\beta_{2,d+3}(\text{Gin}(\bar{I})) > 0,$$

as we desired.

By the above claim and a cancellation principle, R/\bar{I} has a socle element in degree d , and thus R/I has such a socle element in degree d since R/I and R/\bar{I} agree in degrees $\leq d+1$, and hence A cannot be level, as we wished. \square

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let \mathbf{H} and j be as in Theorem 4.4 and let $h_d = d+i$ for $-(d-1) \leq i \leq d+3$.

By Proposition 3.8 in [15], this theorem holds for $-(d-1) \leq i \leq 1$. It suffices, therefore, to prove this theorem for $2 \leq i \leq d+3$. By Theorem 4.4, we have

$$(4.7) \quad \begin{aligned} \beta_{1,d+2}(I^{\text{lex}}) &= \begin{cases} 2, & \text{for } i = 2, \dots, d, \\ 3, & \text{for } i = d+1, d+2, \\ 4, & \text{for } i = d+3, \end{cases} \quad \text{and} \\ \beta_{2,d+2}(I^{\text{lex}}) &= \begin{cases} j+1, & \text{for } i = 2, \dots, d, \\ j+2, & \text{for } i = d+1, d+2, d+3. \end{cases} \end{aligned}$$

Note that if either $j \geq 3$ and $2 \leq i \leq d+3$ or $j = 2$ and $2 \leq i \leq d+2$, then \mathbf{H} is not level since $\beta_{2,d+2}(I^{\text{lex}}) > \beta_{1,d+2}(I^{\text{lex}})$.

Now suppose either $j = 1$ and $2 \leq i \leq d+2$ or $j = 2$ and $i = d+3$. By equation (4.7), we have

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) = \begin{cases} 2, & \text{for } j = 1 \text{ and } i = 2, \dots, d, \\ 3, & \text{for } j = 1 \text{ and } i = d+1, d+2, \\ 4, & \text{for } j = 2 \text{ and } i = d+3. \end{cases}$$

Thus, by Theorem 4.5, \mathbf{H} cannot be level.

It is enough, therefore, to show the case $j = 1$ and $i = d+3$. Assume there exists a level algebra R/I with Hilbert function \mathbf{H} . Applying equation (4.7) again, we have

$$(4.8) \quad \beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) + 1 = 4.$$

Note $h_{d-1} = 2d+4$ and $h_d = h_{d+1} = 2d+3$ in this case. By equation (4.8), the Betti diagram of R/I^{lex} is as follows

| | total: | 1 | - | - | - |
|-------|--------|---|---|---|---|
| <hr/> | | | | | |
| 0: | 1 | - | - | - | |
| 1: | - | - | - | - | |
| | | | | | |
| d-1: | - | * | * | 3 | |
| d: | - | * | 4 | * | |
| d+1: | - | * | * | * | |
| | | | | | |

TABLE 1. Betti diagram of R/I^{lex}

Moreover, by Lemmas 3.8 and 4.2,

$$(4.9) \quad \begin{aligned} \dim_k((I^{\text{lex}} : x_3)/I^{\text{lex}})_d &= \left| \left\{ T \in \mathcal{G}(I^{\text{lex}})_{d+1} \mid x_3 \mid T \right\} \right| \\ &= h_d - h_{d+1} + (h_{d+1})^- \\ &= (h_{d+1})^- \\ &= \left(\binom{d+2}{d+1} + \binom{d+1}{d} \right)^- \\ &= 2. \end{aligned}$$

Hence, using equation (4.9), we can rewrite Table 1 as

| | | | | |
|--------|---|---|---|---|
| total: | 1 | - | - | - |
| <hr/> | | | | |
| 0: | 1 | - | - | - |
| 1: | - | - | - | - |
| | | | | |
| d-1: | - | * | * | 3 |
| d: | - | 2 | 4 | 2 |
| d+1: | - | * | * | * |
| | | | | |

TABLE 2. Betti diagram of R/I^{lex}

Let $J := (I_{\leq d+1})^{\text{lex}}$. Note I^{lex} and J agree in degree $\leq d+1$. Hence we can write the Betti diagram of R/J as

| | | | | |
|--------|---|---|---|---|
| total: | 1 | - | - | - |
| <hr/> | | | | |
| 0: | 1 | - | - | - |
| 1: | - | - | - | - |
| | | | | |
| d-1: | - | * | * | 3 |
| d: | - | 2 | 4 | 2 |
| d+1: | - | a | b | * |
| | | | | |

TABLE 3. Betti diagram of R/J

Since R/I is level and $(I_{\leq d+1})$ has no generators in degree $d+2$, we have $a = 0$ or 1 .

Case 1. Let $a = 0$. Then, by Theorem 2.4, we have $b = 0$. Since J and $(I_{\leq d+1})$ agree in degree $\leq d+1$,

$$\beta_{2,d+3}(J) = \beta_{2,d+3}((I_{\leq d+1})) = 2.$$

This means $R/(I_{\leq d+1})$ has two dimensional socle elements in degree d , so does R/I , which is a contradiction.

Case 2. Let $a = 1$, then J has one generator in degree $d+2$. By Lemmas 3.8 and 4.2,

$$(4.10) \quad \begin{aligned} \dim_k((J : x_3)/J)_{d+1} &= \left| \{ T \in \mathcal{G}(J)_{d+2} \mid x_3 \mid T \} \right| \\ &= h_{d+1} - h_{d+2} + (h_{d+2})^- \end{aligned}$$

where $h_{d+2} = \mathbf{H}(R/J, d+2) = h_{d+1}^{(d+1)} - 1 = (2d+3)^{(d+1)} - 1 = 2d+4$. Hence we obtain $(h_{d+2})^- = (2d+4)^- = 1$, and by equation (4.10)

$$\dim_k((J : x_3)/J)_{d+1} = 0.$$

Applying Theorem 2.4 again, we find

$$b = \beta_{1,d+3}(J) = \sum_{T \in \mathcal{G}(J)_{d+2}} \binom{m(T)-1}{1} = 1$$

since $x_1^{d+2} \notin \mathcal{G}(J)_{d+2}$. Thus R/J has at least one socle element in degree d , and so does $R/(I_{\leq d+1})$. Since R/I and $R/(I_{\leq d+1})$ agree in degree $\leq d+1$, R/I has such a socle element, a contradiction, which completes the proof. \square

The following example shows a case where $j = 1$ and $h_d = 2d + 3$ in Theorem 4.1.

Example 4.6. Let I be the lex-segment ideal in $R = k[x_1, x_2, x_3]$ with Hilbert function

$$\mathbf{H} : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 18 \ 17 \ 17 \ 0 \rightarrow .$$

Note that $h_7 = 17 = 2 \times 7 + 3 = 2d + 3$, which satisfies the condition in Theorem 4.1, and $j = h_6 - h_7 = 18 - 17 = 1$. Hence any Artinian algebra having Hilbert function \mathbf{H} cannot be level.

Inverse systems can also be used to produce new level algebras from known level algebras. This method is based on the idea of *Macaulay's Inverse Systems* (see [14] and [26] for details). We want to recall some results from [25]. Actually, Iarrobino shows an even stronger result and the application to level algebras is:

Theorem 4.7 (Theorem 4.8A, [25]). *Let $R = k[x_1, \dots, x_r]$ and $\mathbf{H}' = (h_0, h_1, \dots, h_e)$ be the h -vector of a level algebra $A = R/\text{Ann}(M)$. Then, if F is a generic form of degree e , the level algebra $R/\text{Ann}(\langle M, F \rangle)$ has h -vector $\mathbf{H} = (H_0, H_1, \dots, H_e)$, where, for $i = 1, \dots, e$,*

$$H_i = \min \left\{ h_i + \binom{(r-1)+(e-i)}{(e-i)}, \binom{(r-1)+i}{i} \right\}.$$

The following example is another case of a level O-sequence of codimension 3 of type in equation (1.1) satisfying $h_d = 2d + 4$.

Example 4.8. Consider a level O-sequence $(1, 3, 5, 7, 9, 11, 13)$ of codimension 3. By Theorem 4.7, we obtain the following level O-sequence:

$$(1, 3, 6, 10, 15, 14, 14).$$

Then $14 = 2 \times 5 + 4$, which shows there exists a level O-sequence of codimension 3 of type in equation (1.1) when $h_d = 2d + 4$.

In general, we can construct a level O-sequence of codimension 3 of type in equation (1.1) satisfying $h_d = 2d + 4$ for every $d \geq 5$ as follows.

Proposition 4.9. *There exists a level O-sequence of codimension 3 of type in equation (1.1) satisfying $h_d = 2d + 4$ for every $d \geq 5$.*

Proof. Note that, from Example 4.8, this proposition holds for $d = 5$.

Now assume $d \geq 6$. Consider a level O-sequence $h = (1, 3, 5, 7, \dots, 2d+1, 2d+3)$ where $d \geq 6$. Since

$$\begin{aligned} & \left(h_i + \binom{d+3-i}{d+1-i} \right) - \binom{i+2}{i} \\ &= \left(2i+1 + \frac{(d+3-i)(d+2-i)}{2} \right) - \frac{(i+1)(i+2)}{2} \\ &= \frac{(2+d)(3+d-2i)}{2} \geq 0, \end{aligned}$$

for every $i = 0, 1, \dots, d-3$, we have

$$\begin{aligned} H_i &= \min \left\{ h_i + \binom{d+3-i}{d+1-i}, \binom{i+2}{i} \right\} \\ &= \min \left\{ 2i+1 + \frac{(d+3-i)(d+2-i)}{2}, \frac{(i+1)(i+2)}{2} \right\} \\ &= \frac{(i+1)(i+2)}{2}. \end{aligned}$$

Hence, by Theorem 4.7, we obtain a level O-sequence $\mathbf{H} = (H_0, H_1, \dots, H_d, H_{d+1})$ as follows:

$$\begin{aligned}
H_0 &= 1, \\
H_1 &= 3, \\
&\vdots \\
H_i &= \frac{(i+1)(i+2)}{2}, \\
&\vdots \\
H_{d-2} &= \min \left\{ h_{d-2} + \binom{5}{3}, \binom{d}{d-2} \right\} = \min \left\{ 2d + 7, \frac{(d-1)d}{2} \right\} = 2d + 7, \\
H_{d-1} &= \min \left\{ h_{d-1} + \binom{4}{2}, \binom{d+1}{d-1} \right\} = \min \left\{ 2d + 5, \frac{d(d+1)}{2} \right\} = 2d + 5, \\
H_d &= \min \left\{ h_d + \binom{3}{1}, \binom{d+2}{d} \right\} = \min \left\{ 2d + 4, \frac{(d+1)(d+2)}{2} \right\} = 2d + 4, \\
H_{d+1} &= \min \left\{ h_{d+1} + \binom{2}{0}, \binom{d+3}{d+1} \right\} = \min \left\{ 2d + 4, \frac{(d+2)(d+3)}{2} \right\} = 2d + 4,
\end{aligned}$$

as we wished. \square

Remark 4.10. By the same idea as in the proof of Proposition 4.9, we can construct a level O-sequence of codimension 3 of type in equation (1.1) satisfying

$$2d + (k + 1) = H_{d-1} > H_d = H_{d+1} = 2d + k, \quad (5 \leq k \leq \frac{d^2 - 3d + 2}{2}).$$

For example, if we use

$$h = (1, 3, 6, \dots, 2d + (k - 5), 2d + (k - 3), 2d + (k - 1)),$$

then we construct a level O-sequence of codimension 3 of type in equation (1.1) satisfying

$$\begin{aligned}
H_{d-1} &= \min \left\{ h_{d-1} + \binom{4}{2}, \binom{d+1}{d-1} \right\} = \min \left\{ 2d + (k + 1), \frac{d(d+1)}{2} \right\} = 2d + (k + 1), \\
&(\because k \leq \frac{d^2 - 3d - 2}{2}), \\
H_d &= \min \left\{ h_d + \binom{3}{1}, \binom{d+2}{d} \right\} = \min \left\{ 2d + k, \frac{(d+1)(d+2)}{2} \right\} = 2d + k, \\
H_{d+1} &= \min \left\{ h_{d+1} + \binom{2}{0}, \binom{d+3}{d+1} \right\} = \min \left\{ 2d + k, \frac{(d+2)(d+3)}{2} \right\} = 2d + k,
\end{aligned}$$

as we desired.

Using Theorem 4.1, we know that some non-unimodal O-sequence of codimension 3 cannot be level as follows.

Corollary 4.11. Let $\mathbf{H} = \{h_i\}_{i \geq 0}$ be an O-sequence with $h_1 = 3$. If

$$h_{d-1} > h_d, \quad h_d \leq 2d + 3, \quad \text{and} \quad h_{d+1} \geq h_d$$

for some degree d , then \mathbf{H} is not level.

Proof. Note that, by the proof of Theorem 4.1, any graded ring with Hilbert function

$$\mathbf{H}' : h_0 \ h_1 \ \cdots \ h_{d-1} \ h_d \ h_d \rightarrow$$

has a socle element in degree $d - 1$.

Now let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with Hilbert function \mathbf{H} . If $A_{d+1} = \langle f_1, f_2, \dots, f_{h_{d+1}} \rangle$ and $I = (f_{h_d+1}, \dots, f_{h_{d+1}}) \bigoplus_{j \geq d+2} A_j$, then a graded ring $B = A/I$ has Hilbert function

$$h_0 \ h_1 \ \cdots \ h_{d-1} \ h_d \ h_d.$$

Hence B has a socle element in degree $d - 1$ or d by Theorem 4.1. Since $A_i = B_i$ for every $i \leq d$, A also has the same socle element in degree $d - 1$ or d as B , and thus \mathbf{H} is not level as we wished. \square

The following is an example of a non-level and non-unimodal O-sequence of codimension 3 satisfying the condition of Corollary 4.11.

Example 4.12. Consider an O-sequence

$$\mathbf{H} : 1 \ 3 \ 6 \ 10 \ 15 \ 20 \ 18 \ 17 \ h_8 \ \dots$$

There are only 3 possible O-sequences such that $h_8 \geq h_7 = 17$ since $h_8 \leq h_7^{(7)} = 17^{(7)} = 19$. By Theorem 4.1, \mathbf{H} is not level if $h_8 = h_7 = 17$. The other two non-unimodal O-sequences, by Corollary 4.11,

$$\begin{array}{ccccccccccccc} 1 & 3 & 6 & 10 & 15 & 20 & 18 & 17 & 18 & 19 & \dots \\ 1 & 3 & 6 & 10 & 15 & 20 & 18 & 17 & 19 & \dots \end{array}$$

cannot be level either.

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